Supporting engineering students learning mathematical induction with an online tutorial

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ABSTRACT

We describe an online tutorial that was developed in order to support first year engineering students' learning about mathematical induction (MI). The tutorial integrates theoretical explanations, examples and interactive reflective questions, and was designed to increase students' engagement by creating frequent interactions and using a varied collection of reflective questions. The tutorial was developed according to research-based knowledge concerning students' difficulties with MI and considering global vs. local proof comprehension. We examined the effects of the MI tutorial on the following students' achievements: (i) students' grade in the final quiz of the tutorial (FTG); (ii) students' grade in the MI question in the final exam of the course. We collected students' initial/final quiz-grades (ITG, FTG), the time students worked on the tutorial, the number of final quiz trials and students' grades in the MI question in the final exam in five semesters (before/after incorporating the tutorial). Our findings indicate that the mean FTG is significantly higher than the mean ITG (e.g., in the first semester, N=152, mean ITG=34.5; mean FTG=73.2). Apparently, the instructional part of the tutorial had a positive short-term effect on students' FTG. However, we did not find a major effect of the MI tutorial on students' grade in the MI exam question (regardless of the type of claims to be proved and other circumstantial exam settings). We also found that most students answer the MI question in the exam, which may suggest that students believe that they understand the use of MI; yet, their mean grade in this question is not very high (51.7-68.8). In addition, a change in course policy (including the FTG in the course's final grade), motivated students to achieve a high FTG but the time that students worked on the tutorial decreased, which may explain the lack of long-term effect.

KEYWORDS

Proof teaching, Mathematical induction, Online tutorial, Tertiary mathematics, Standards: 2,8,11

BACKGROUND

Mathematics is considered a fundamental subject in engineering education, since mathematical skills involve logical thinking, problem solving abilities and enable high achievements in other engineering subjects; In addition low performance in mathematical courses inhabit an academic risk and influence students' motivation (González et al., 2020).

There is an increasing interest in mathematics teaching practices at the tertiary level and in alternative approaches to mathematics teaching other than lecturing. Few of the subjects that are being studied are teaching and learning of mathematical proofs, effective ways of teaching mathematics to non-mathematics students and students' use of online resources (Biza et al., 2016).In addition, scholars are contemplating about different forms of assessment of students' knowledge and what type of performances should be confirmed in the assessment (Bennedsen, 2021). The use of online assessment is another strand of investigation that is gaining more attention, in particular in recent times (e..g., Pick and Cole, 2021).

In this paper, we describe an online tutorial developed by two of the authors, designed to help first year engineering students learning about proof by mathematical induction (MI), and discuss the effects that the MI online tutorial had on different aspects of students' achievements and learning.

Teaching and learning proof at the tertiary level

Research has identified a vast list of cognitive difficulties related to proving at all levels, including the tertiary level, for example: a lack of acquaintance with proving strategies, difficulties with mathematical language and notation, knowing how to work with mathematical definitions and understanding the logical structure of a proof (Selden & Selden, 2008, 2013). Non-mathematics students are expected to improve their proof constructing ability throughout their mathematics courses, which focus on a deep understanding of mathematical content but not necessarily on the concept of proof itself (Selden & Selden, 2008). Researchers have also been giving growing attention to affective aspects that influence the learning of mathematics, acknowledging their strong effect on students' proving process and problem solving abilities (Selden & Selden, 2013). In spite of these difficulties, there is an agreement among mathematics education researchers and mathematics lecturers that reasoning and proving are central both to knowledge construction and to the establishment of a mathematical community in the classroom. Research about teaching proof at the tertiary level did not yet accomplish a solid corpus of well-established ways for proof teaching, but a few approaches were suggested and studied in the literature. We focus here on two such approaches.

Firstly, Alcock (2009) designed a computer-based presentation of proofs called 'e-proofs', aimed to make the proof's structure and reasoning more explicit and visible to students. An e-proof comprised of a set of slides, showing a theorem and its complete proof, accompanied with audio commentary containing explanations similar to those a lecturer would give in a frontal lecture. Alcock et al. (2015) compared the effects of e-proofs with two other proof presentations (a frontal lecture, written proof) on undergraduate students' proof comprehension and found that although students liked e-proofs and perceived them as helpful, e-proofs were less desirable than textbook proofs or frontal lecture in terms of proof understanding. Alcock et al. speculated that e-proofs helped students' on-spot understanding without investing too much effort, which caused the lesser sustainability of their understanding. In fact, Alcock (2009) related to similar concerns stating that although e-proofs allow the teacher to better articulate their own understanding of a proof, students' interactivity is low and is mainly expressed by controlling pace and order of content. Alcock et al. further concluded that students' self-explanation training improves both students' mathematical reading and proof comprehension.

The second approach is the 'proof framework' instruction (Selden & Selden, 2013), designed to help students develop proof competencies. The term 'proof framework' relates to the formal-rhetorical part of the final written proof, which depends on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. Selden and Selden let students prepare proof frameworks, leaving blanks in the proofs that should be filled with mathematical problem-solving content, and claim that writing such frameworks "…

not only improves their proof writing… but also… can reveal the nature of the problem(s) to be solved…" (p. 309). Selden and Selden reported that although constructing proof frameworks might be challenging for students with little experience in proof writing, after practice it can become routine and improve students' proof writing according to accepted community norms.

Stylianides and Stylianides (2017) asserted that although mathematics education research identified many difficulties in teaching and learning proof and suggested alternative pedagogical methods, less research focused on designing interventions and examining their effects on learning proof; they recommend applying research-based interventions in the mathematics classroom, or in any other formal learning setting. However, how can one assess the effect of such an intervention on students' proof comprehension? Mejia-Ramos et al. (2012) presented an assessment model for undergraduate students' proof comprehension, which may be used to design assessment instruments of students' proof understanding, as well as to evaluate the effectiveness of a specific mathematics instruction. The model addresses local and global proof comprehension. Local proof comprehension relates, for example, to understanding a specific statement and how it connects to a small number of other statements, the definition of terms and identifying the specific data supporting a claim. Global proof comprehension relates, for example, to the proof as a whole entity, and to aspects such as being able to reflect on main ideas, breaking the proof into modules and identifying the logical relation between them, applying the method of the proof in other contexts and choosing suitable illustrative examples.

Teaching and learning mathematical induction

Mathematical induction (MI) is a proving method frequently employed by mathematicians. There are different formulations of proof by MI and we use the following: Suppose one wishes to prove a conjecture that a statement $P(n)$ holds for all $n \in N$. Proof by MI has three steps:

i. The inductive base: prove that $P(1)$ holds;
ii. The inductive assumption: assume $P(k)$ holds for some $k \in N$;
iii. The inductive step: prove that $P(k) \Rightarrow P(k+1)$.

The conclusion is that $P(n)$ holds for all $n \in N$.

The pedagogical importance of teaching and learning MI in secondary school and in college was already discussed by Young (1908), who claimed that "the process of mathematical induction is exceptionally well fitted to introduce the beginner to the philosophic study of mathematical thinking" (p. 146). More than a century later, Stylianides et al. (2016) investigated the explanatory potential of proving by MI and suggested that "the explanatory power of proving by mathematical induction can help students develop their understanding of … mathematical ideas,…, ideas about proof, or both" (p. 23). Engineering students learn MI as part of their basic mathematical education, since in addition to being a fundamental and powerful proving method, it develops the logical thinking required for engineering and develops students' ability to work with sequences, particularly recursive sequences. This is true for all engineering students but bears particular importance for software engineering students, as Gunderson (2010) explains: "…because of the recent explosion of knowledge in combinatorics, computing, and discrete mathematics, mathematical induction is now, more than ever, critical in education…The theory of recursion in computing science is practically the study of mathematical induction applied to algorithms. The theory of mathematical logic and model theory rests entirely on mathematical induction, as does set theory... mathematical induction is absolutely essential in linear algebra, probability theory, modelling, and analysis…" (p. xix).

However, researchers have documented many difficulties that students encounter while learning MI, and we refer to a few main ones. Firstly, students think that MI is a circular proof in which they assume what they are trying to prove; this reflects a deep misunderstanding of the structure of proof by MI (Ernest, 1984). Secondly, many university students believe that proof by MI is a technical and superficial way of proving and do not understand the structure of the proof, in particular that the inductive base and the inductive step are independent and are both necessary for a valid proof (Movshovitz-Hadar, 1993; Ron & Dreyfus, 2004; Stylianides, Sandefur & Watson, 2016). Finally, students do not perceive MI as a natural development of their previous mathematical experience but as detached from other topics (Ernest, 1984). In order to tackle some of these difficulties, mathematics educators offered various recommendations, for example using cognitive conflict to stress the necessity and independence of the inductive base and the inductive step (Ernest, 1984; Movshovitz-Hadar, 1993) or using models such as the Domino tiles model (Ron & Dreyfus, 2004).

RATIONALE AND OBJECTIVES

We situate our work within the growing research field concerned with mathematics teaching practices at the tertiary level, in particular online tutorials that support the teaching of proof. Our research is an intervention, aimed to examine the effects of a specially designed online tutorial about MI on students' learning. MI was chosen because in spite of its centrality as a proving method in mathematics, it is usually not taught in secondary mathematics and most of our engineering students encounter it for the first time. In addition, MI has a clear structure or 'proof framework' (Selden & Selden, 2013), and there is vast established knowledge about students' difficulties with MI.

The online MI tutorial started with a quiz and proceeded with an instructional part containing theoretical explanations, examples and interactive reflective questions designed to support global and local proof comprehension (Mejia-Ramos et al., 2017) and to increase students' involvement. The tutorial ended with a quiz and the students received a final tutorial grade (FTG).

The objectives of the study presented here are to examine the effects of the instructional part of the MI tutorial on students' FTG and on students' grade in the MI question in the final exam of the course.

METHOD

The study was conducted in a discrete mathematics course, taken by Software engineering students and Industrial engineering students in an Engineering college in Israel. The course is taken by students in their first year of studies. As stated above, most students did not learn proof by MI in high-school; all students use MI in other courses (e.g., Calculus). In the course, MI is taught in a frontal lesson (3 hours). The lesson was supported by an online tutorial designed and programmed (using Articulate Storyline - an application used to build interactive online courses) by two lecturers of the course. The tutorial starts with a quiz that is graded (ITG), but the students do not get the grade or any feedback. It proceeds with an elaborated instructional part, and ends with a final quiz, identical to the initial one, which the students can repeat, receive feedback and a final tutorial grade (FTG). The initial/final quiz contains 10 questions that relate to global and local comprehension of proof by MI.

The instructional part of the tutorial is divided into sections, some contain theoretical explanations about MI together with examples and some MI proofs of different types of claims (algebraic, geometric). Figure 1 presents a scheme of the MI tutorial. The tutorial is interactive; students answer different types of reflective questions (e.g., typing algebraic expressions, multiple-choice, dragging expressions), that require global or local proof comprehension; the tutorial continues when students answer correctly. Students can also return to a previous section using a side content.

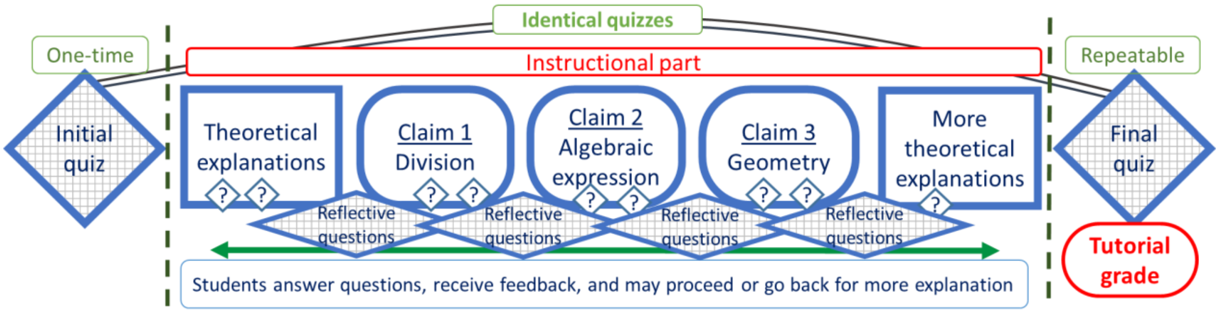

Figure 1: A schematic representation of the MI tutorial

Figure 2 presents an example of two tutorial screens containing global/local questions. The students have to complete the tutorial at home in their own pace, as long as they complete it before the end of the semester.

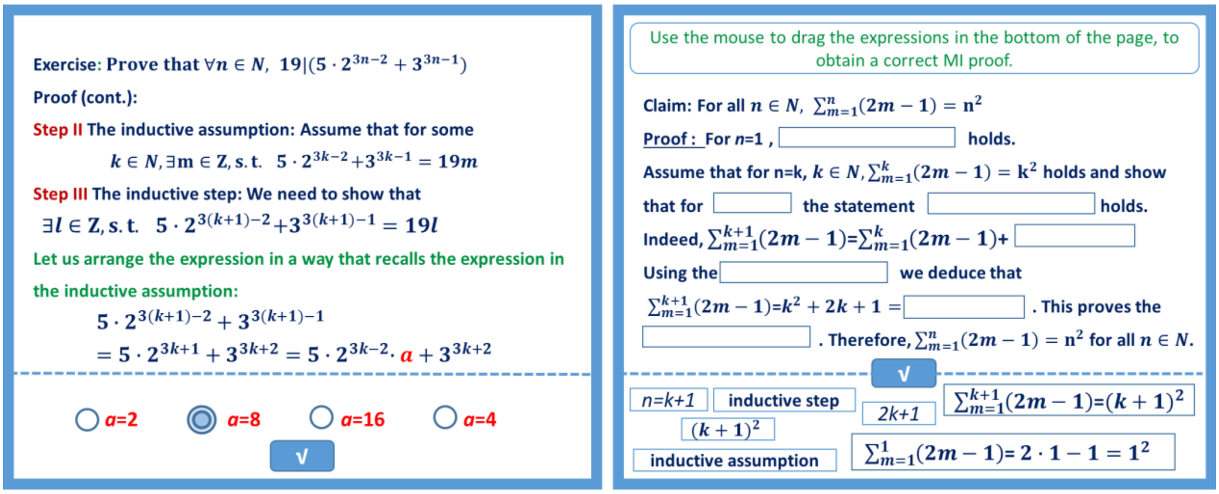

Figure 2: Examples for reflective questions (left/right – local/global comprehension)

Every final exam in discrete mathematics course includes a question (or part of a question) about proof by MI and the students have a choice of overall 5 of 6 questions. We collected data from two semesters before the incorporation of the MI tutorial in the course (Sem-BT1, Sem-BT2) and three semesters after the MI tutorial was incorporated as a mandatory activity in the course (Sem-T, Sem-TG1, Sem-TG2). In Sem-T the students were simply required to finish the tutorial before the final exam. In Sem-TG1/Sem-TG2 the FTG was incorporated in the final grade of the course (range 0-100): the students received 2.5 points in the final grade of the course, only if FTG ≥60.

We collected students' grades in the MI question in all five semesters. In the semesters that the MI tutorial was incorporated, we collected students' grades in initial and final tutorial quizzes (ITG/FTG), the time that students worked on the tutorial and the number of final quiz trials. Table 1 presents the MI final exam questions in each semester.

Table 1: Final exam MI questions

Semester	MI question in the final exam
Sem-BT1	Prove by MI that for any $n \in N$, $4^n + 15n - 1$ is divisible by 9
Sem-BT2	Prove by MI that for any $n \in N$, $\sum_{k=1}^{n} k^2 < \frac{(n+1)^3}{3}$
Sem-T	We will prove by MI that $\forall n \in N,\ 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ (a) Check the inductive base; (b) Write explicitly the inductive assumption and what is needed to prove in order to show that the inductive step holds; (c) Prove the inductive step; (d) Write explicitly the conclusion; (e) prove that $\forall n \in N,\ 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2$; (f) Can the claim in (e) be proven by MI? Please explain.
Sem-TG1	Prove by MI that $\forall n \in N,\ \sum_{j=1}^{n} (2j-1)^2 = \frac{n(4n^2-1)}{3}$
Sem-TG2	Prove by MI that $\forall n \in N,\ 4^{n+1} + 5^{2n-1}$ is divisible by 21

FINDINGS

Students' achievements in MI tutorial

Table 2 (below) presents the mean of students' grades in the initial/final quiz (ITG/FTG) in three semesters. The two columns on the right present the percentage of students who repeated the tutorial, where the term 'successful repeating students' (the last column on the right) relates to students that their first FTG was ≥ 60 but repeated the tutorial nevertheless.

We considered data of students with FTG; we omitted data of students with FTG=0 where the time they worked on the tutorial was less than 2 minutes or more than two hours. Mean time was calculated for students that worked on the tutorial and completed the first trial of the quiz in less than 2 hours.

Table 2: Tutorial grades and other tutorial parameters (SD = Standard deviation)

	ITG Mean (SD)	FTG Mean (SD) First trial	FTG Mean (SD) Highest trial	Mean time of first trial [min.]	Mean number of trials	% of repeating students	% of successful repeating students
Sem-T (N=152)	34.54 (20.3)	73.2 (18.3)	74.47 (18.8)	42.41	1.05	4	4
Sem-TG1 (N=186)	38.06 (20.2)	65.54 (23.6)	82.26 (16.5)	30:28	1.51	41	13
Sem-TG2 (N=169)	36.09 (21.9)	53.31 (34.3)	85.56 (15.9)	26:03	1.89	56	9

Students' achievements in final exam

Table 3 presents the MI question grade statistics and the percentage of students that answered the MI exam question in each semester. In Sem-T* we calculated mean grade for items a-d of the MI question (omitting the grades of items e-f), so that the question is more similar to MI exam questions in other semesters (Table 1) and the grades will be more comparable. The separate grades of items a-f are presented in Table 5.

Table 3: Students' grade (0-100) in MI question in final exam (SD = Standard deviation)

	Sem-BT1 (N=113, 72%)	Sem-BT2 (N=137, 94%)	Sem-T* (N=159, 92%)	Sem-TG1 (N=171, 96%)	Sem-TG2 (N=145, 95%)
Mean (SD)	51.7 (34.1)	68.8 (29.2)	60 (27.2)	63.3 (23.5)	67.6 (36.8)
Median	40	80	55	66.7	100

In order to investigate further the relation between students' first/highest FTG and students' grade in the MI final exam question we calculated the correlations between these grades, as presented in Table 4. We regarded only to grades of students who have a FTG and answered the MI exam question.

Table 4: Correlation between first/highest tutorial grade and grade in MI exam question

	Sem-T (N=124)	Sem-TG2 (N=153)	Sem-TG1(N=138)
Correlation	-0.01 / 0.04	0.19/ -0.1	-0.07 / -0.01

Finally, Table 5 presents students' grades in items a-f in the MI exam question in Sem-T (see items a-f in Table 1).

Table 5: Students' mean grade breakdown (0-100) in MI question in final exam in Sem-T

N=159	a (base)	b (assumption)	c (step)	d (conclusion)	e (consequence)	f (proof comprehension)
Mean grade	90.57	63.52	51.70	64.15	44.65	24.21

DISCUSSION

Our first objective was to study the effects of the instructional part of the MI tutorial on students' final tutorial grade (FTG). Table 2 demonstrates that the mean FTG after the first trial of the final quiz is significantly higher than the mean ITG. It seems that the instructional part of the MI tutorial had a very positive effect on students' FTG. The other tutorial parameters that we examined (Table 2) demonstrate that the mean time for completing the tutorial and the mean FTG after first trial decreased; the mean highest FTG as well as the mean of number of trials increased. There is also a big gap between the percentage of students who repeated the tutorial in Sem-T and Sem-TG1/Sem-TG2. In other words, in Sem-TG1 and Sem-TG2 students spent less time on their first trial of the tutorial, their first FTG decreased, but they repeated the final quiz until they achieved a higher FTG. We suspect that this is a result of the change in course grading policy, for as explained above in Sem-TG1 and Sem-TG2 the course grading policy changed and the FTG became an ingredient in the final grade, probably motivating students to achieve higher FTG. However, Table 2 also demonstrates that in all the semesters there were students that repeated the tutorial even though they already gained a passing grade, i.e their first FTG was ≥ 60. This happened regardless of the fact that the FTG itself was not part of the course's grade (as explained above students received 2.5 points in the final grade of the course if FTG ≥60). This may point to high motivation of these students and a high level of engagement with the tutorial. Pick and Cole (2021) report similar phenomenon in their study, concerning students that attempted to increase their score even after a pass mark had been achieved. If this is a feature of using online tutorials and quizzes – it is a very positive one, and should be further investigated.

Our second objective was to study the effect of the MI tutorial on students' grade in the MI exam questions. Table 3 does not demonstrate a clear effect of the MI tutorial neither on the mean nor on the median in the MI question. In fact, we did not detect any clear trend in the grades of the MI question along the five semesters. Of course, one should consider the difference in the type of claims to be proved (sum, division, etc.), the questions' type (with/without division into items) and even who the grader of the question was (the exams are checked each semester by a different lecturer). In addition, some of the semesters in which the data was collected were during the Covid-19 pandemic so the external circumstances varied from semester to semester (class exams, home exams). If we examine students' grades in Sem-BT1 and Sem-TG2, in which the exam questions were very similar (proving claims about division properties), the data shows that the mean and median grade are higher in Sem-TG2 but it is difficult to deduce that the increase was a result of the incorporation of the MI tutorial. The data in Table 4 supports the lack of a major effect of the MI tutorial on students' grade in the MI exam question. Nevertheless, Table 3 reflects that the percentage of students that choose to answer the MI question is high in all semesters (in 4 of 5 semesters it exceeds 90%). This suggests that students possess high beliefs concerning their ability to prove claims using MI, in spite of the fact that their mean grade is not very high (51.7-68.8). This supports research that asserted that students (at various levels) perceive proof by MI as a series of technical manipulations and do not possess a deep understanding of the structure and logic of MI (e.g., Ron & Dreyfus, 1994).

In order to consider other effects of the MI tutorial we relate to the MI exam question in Sem-T. Items a-d of the question resembled MI questions in other final exams; items e-f required less procedural understanding and involved meta-mathematical thinking. Table 4 demonstrates that students encountered difficulties especially in item c and in items e-f. Based on our experience, the difficulties in item c had mainly a technical nature and concern performing algebraic manipulations. Yet, the grades in items e-f indicate that students were unable to deduce a direct conclusion from the claim they have just proved; they were also unable to explain why they cannot use MI to prove a slightly different claim, where such an explanation requires local proof comprehension of the MI proof they have just performed in items a-d. In that sense, it seems that the MI tutorial did not support profound long-term proof comprehension. Granted, the MI tutorial did not focus on enhancing students' understanding of such subtleties. We consider this a matter for future research, in particular how to improve the tutorial to support deeper students' understanding.

CONCLUSIONS

To conclude, one aim of the MI tutorial was to teach students how to construct and write a correct MI proof, in the sense that they will be able to construct a correct 'proof framework' (Selden & Selden, 2013). It seems that the use of the MI tutorial had a positive effect on short-term students' grades (FTG) but no clear effect on their MI exam question grade. This finding is similar to what Alcock et al. (2015) called the 'on-spot' effect of e-proofs, which Alcock (2009) regarded a consequence of students' relatively passive learning. The design of the MI tutorial presented in this paper took this into account and encouraged students' activity by creating frequent interactions and using a varied collection of reflective questions, yet we did not notice a clear positive effect of the MI tutorial on students' achievements in the MI exam question. Thus, our main pedagogical conclusion is that relying solely on an online tutorial to address students' difficulties with MI is an unrealistic expectation and that the online tutorial cannot replace a discussion in a frontal lecture.

However, we still did not address possible effects of the MI tutorial on affective aspects of learning, such as learning experience or motivation and on their learning habits. In our study, we have overall positive feedback from students regarding these aspects, in concurrence with other studies that examine the use of online learning materials and quizzes, e.g., Pick and Cole (2021), who report students' high satisfaction with online quizzes. González et al. (2020) found that developing metacognition skills, time management and study habits help students to overcome challenges in their engineering studies and concluded that engaging engineering students in new learning spaces supports the development of these skills. Mathematical online tutorials, as the MI tutorial we discuss, are an example of such new learning spaces. Yet, reaching established conclusions on this matter requires further research.

Other future research directions concern the effects of flipping the MI lesson: replacing the frontal lecture by using the MI tutorial as a self-study unit, discussing MI in class and repeating the study while maintaining higher standardization (e.g., regarding exam questions and grading policy). We believe that as Stylianides and Stylianides (2017) recommended, intervention studies are an important source for gaining information of effective teaching methods, especially in times when online teaching is becoming more common.

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